Partial Solution Set, Leon §4.3

4.3.2 Let $[\mathbf{u}_1, \mathbf{u}_2]$ and $[\mathbf{v}_1, \mathbf{v}_2]$ be ordered bases for $\mathbf{R}^2$, where $\mathbf{u}_1 = (1,1)^T$, $\mathbf{u}_2 = (-1,1)^T$, $\mathbf{v}_1 = (2,1)^T$, and $\mathbf{v}_2 = (1,0)^T$. Let L be the linear transformation defined by $L(\mathbf{x}) = (-x_1, x_2)^T$, and let B be the matrix representing L with respect to $[\mathbf{u}_1, \mathbf{u}_2]$. {Note: B was actually part of problem 1 in this chapter. As usual, the first column of B is $[L(\mathbf{u}_1)]_U = (0,1)^T$, and the second column of B is $[L(\mathbf{u})_2]_U = (1,0)^T$.}

(a) Find the transition matrix S corresponding to the change of basis from $[\mathbf{u}_1, \mathbf{u}_2]$ to $[\mathbf{v}_1, \mathbf{v}_2]$.

Solution: The transition matrix in question is the one I've been calling T_{UV}, i.e.,

$$S = V^{-1}U = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}.$$

(b) Find the matrix A representing L with respect to $[\mathbf{v}_1, \mathbf{v}_2]$ by computing $A = SBS^{-1}$.

Solution: First we find $S^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$. Then it is a simple matter to determine that $A = SBS^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}$.

4.3.3 Let L be the linear transformation on $\mathbf{R}^3$ given by

$$L(\mathbf{x}) = (2x_1 - x_2 - x_3, 2x_2 - x_1 - x_3, 2x_3 - x_1 - x_2)^T,$$

and let A be the matrix representing L with respect to the standard basis for $\mathbf{R}^3$. If $\mathbf{u}_1 = (1,1,0)^T$, $\mathbf{u}_2 = (1,0,1)^T$, and $\mathbf{u}_3 = (0,1,1)^T$, then $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ is an ordered basis for $\mathbf{R}^3$.

(a) Find the transition matrix U corresponding to the change of basis from $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ to the standard basis.

(b) Determine the matrix B representing L with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

Solution:

(a) This is simply $U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(b) Somewhat surprisingly, $B = U^{-1}AU = A$. An interesting sidelight: this means that $UA = AU$, i.e., we have an instance of a commuting pair of matrices.

4.3.4 Let L be the linear operator mapping $\mathbf{R}^3$ into $\mathbf{R}^3$ defined by $L(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}$. Let $\mathbf{v}_1 = (1,1,1)^T$, $\mathbf{v}_2 = (1,2,0)^T$, and $\mathbf{v}_3 = (0,-2,1)^T$. Find the transition matrix V corresponding to a change of basis from $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to the standard basis, and use it to determine the matrix B representing L with respect to $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$.

Solution: The transition matrix is $V = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$. We want

$$B = V^{-1}AV = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4.3.5 Let L be the linear operator on P_3 defined by

$$L(p(x)) = xp'(x)\text{''} + p''(x).$$

(a) Find the matrix A representing L with respect to $[1, x, x^2]$.

(b) Find the matrix B representing L with respect to $[1, x, 1 + x^2]$.

(c) Find the matrix S such that $B = S^{-1}AS$.

(d) Given $p(x) = a_0 + a_1 x + x_2(1 + x^2)$, find $L^n(p(x))$.

Solution:

(a) We start by applying L to the basis vectors: $L(1) = 0$, $L(x) = x$, and $L(x^2) = 2x^2 + 2$. The corresponding coordinate vectors become the columns of $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) The coordinate vectors for 1 and x are unchanged, but the coordinate vector for $2x^2 + 2$ is now $(0, 0, 2)^T$, so $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

(c) The change of basis matrix has for its columns the coordinate vectors of the basis from part (b): $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(d) The coordinate vector of $p(x)$ is $(a_0, a_1, a_2)^T$. The nth power of B is simple to compute because of the simple diagonal structure of B; $B^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{bmatrix}$. It follows that the coordinate vector for $L^n(p(x))$ is $B^n(a_0, a_1, a_2)^T = (0, a_1, 2^n a_2)$, so $L^n(p(x)) = a_1 x + 2^n a_2(1 + x^2)$.

4.3.8 Suppose that $A = S\Lambda S^{-1}$, where Λ is a diagonal matrix with main diagonal $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Show that $A\mathbf{s}_i = \lambda_i \mathbf{s}_i$ for each $1 \le i \le n$.

(b) Show that if $\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{s}_i$, then $A^k\mathbf{x} = \sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{s}_i$.

(c) Suppose that $|\lambda_i| < 1$ for each $1 \le i \le n$. What happens to $A^k\mathbf{x}$ as $k \to \infty$?

Solution:

(a) For any choice of i, $1 \le i \le n$, we have

$$\begin{aligned} A\mathbf{s}_i &= \left(S\Lambda S^{-1}\right)\mathbf{s}_i \\ &= S\Lambda\left(S^{-1}\mathbf{s}_i\right) \\ &= S\Lambda\mathbf{e}_i \\ &= S\left(\Lambda\mathbf{e}_i\right) \\ &= S\lambda_i\mathbf{e}_i \\ &= \lambda_i S\mathbf{e}_i \\ &= \lambda_i\mathbf{s}_i. \end{aligned}$$

(b) This is easily proven by induction: $A^0\mathbf{x} = \mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{s}_i$.

Assume that $A^k\mathbf{x} = \sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{s}_i$ for some $k \in \mathbf{N}$. Then

$$\begin{aligned} A^{k+1}\mathbf{x} &= AA^k\mathbf{x} \\ &= A\sum_{i=1}^{n} \alpha_i\lambda_i^k\mathbf{s}_i \\ &= \sum_{i=1}^{n} A\alpha_i\lambda_i^k\mathbf{s}_i \\ &= \sum_{i=1}^{n} \alpha_i\lambda_i^k A\mathbf{s}_i \\ &= \sum_{i=1}^{n} \alpha_i\lambda_i^k\lambda_i\mathbf{s}_i \\ &= \sum_{i=1}^{n} \alpha_i\lambda_i^{k+1}\mathbf{s}_i, \end{aligned}$$

and the result follows by induction. $\square$

(c) Each term in the preceding sum vanishes, since if $|\lambda| < 1$ then $\lim_{k\to\infty} \lambda^k = 0$.

4.3.9 Suppose that $A = ST$, where S is nonsingular. Let $B = TS$. Show that B is similar to A.

Proof: Assume that A is as described, i.e., that $A = ST$ and that S is nonsingular. Then

$$B = TS = (S^{-1}S)TS = S^{-1}(ST)S = S^{-1}AS,$$

so B is similar to A. □

What's the point? Given any square S and T, with at least one of the two nonsingular, we know that it's unlikely that $ST = TS$. But at least ST and TS are similar. And that (as we shall see) means that they have much in common (eigenvalues, for example).

4.3.10 Let A and B be $n \times n$ matrices. Show that if A is similar to B, then there exist $n \times n$ matrices S and T, with S nonsingular, such that $A = ST$ and $B = TS$.

Solution: Well, at least a hint. Note that we are proving the converse of (9). This is perhaps easier than it initially seems. Assume that A is similar to B. You may then write B in terms of A and another (nonsingular) matrix S, right? Do so. Now what?